

TIGHT FIBERED KNOTS AND BAND SUMS

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ABSTRACT. We give a short proof that if a non-trivial band sum of two knots results in a tight fibered knot, then the band sum is a connected sum. In particular, this means that any prime knot obtained by a non-trivial band sum is not tight fibered. Since a positive L-space knot is tight fibered, a non-trivial band sum never yields an L-space knot. Consequently, any knot obtained by a non-trivial band sum cannot admit a finite surgery.

For context, we exhibit two examples of non-trivial band sums of tight fibered knots producing prime knots: one is fibered but not tight, and the other is strongly quasipositive but not fibered.

1. INTRODUCTION

Let $K_1 \sqcup K_2$ be a 2-component split link in the 3-sphere S^3 . Let $\beta: [0, 1] \times [0, 1] \rightarrow S^3$ be an embedding such that $\beta([0, 1] \times [0, 1]) \cap K_1 = \beta([0, 1] \times \{0\})$ and $\beta([0, 1] \times [0, 1]) \cap K_2 = \beta([0, 1] \times \{1\})$. Then we obtain a knot $K_1 \natural_{\beta} K_2$ by replacing $\beta([0, 1] \times \{0, 1\})$ in $K_1 \sqcup K_2$ with $\beta(\{0, 1\} \times [0, 1])$. We call $K_1 \natural_{\beta} K_2$ a *band sum* of K_1 and K_2 with the band β . In the following, for simplicity, we use the same symbol β to denote the image $\beta([0, 1] \times [0, 1])$. We say that a band sum is *trivial* if one of K_1 and K_2 , say K_2 , is the unknot and the band β gives just a connected sum. If the band sum is trivial, then obviously $K_1 \natural_{\beta} K_2 = K_1$. The converse also holds, i.e. if $K_1 \natural_{\beta} K_2 = K_1$, then the band sum is trivial; see [3, 19]. A band sum is regarded as a natural generalization of the connected sum, and many prime knots are obtained by band sums; see [8].

We say a fibered knot in S^3 is *tight* if, as an open book for S^3 , it supports the positive tight contact structure on S^3 . A knot in S^3 is *strongly quasipositive* if it is the boundary of a *quasipositive* Seifert surface, a special kind of Seifert surface obtained from parallel disks by attaching positive bands in a particular kind of braided manner, for a more precise definition see e.g. [18, 61.Definition]. It is shown by Rudolph [17, Characterization Theorem] (cf. [18, 90.Theorem]) that a knot K is strongly quasipositive if and only if it has a Seifert surface that is a subsurface of the fiber of some positive torus knot. Hedden proved that tight fibered knots are precisely the fibered strongly quasipositive knots [5, Proposition 2.1]; Baader-Ishikawa [1, Theorem 3.1] provides an alternative proof. From this correspondence, one may observe that a connected sum of two tight fibered knots is again a tight fibered knot. (Of course this follows more directly from the definitions of connected sums of contact manifolds and contact structures supported by open books, e.g. [2, 4].)

The aim of this note is to prove:

Theorem 1.1. *If a tight fibered knot in S^3 is a non-trivial band sum, then the band sum expresses the knot as a connected sum. In particular, it is not prime.*

Corollary 1.2. *If a prime knot in S^3 is a non-trivial band sum, then it is not a tight fibered knot.*

A knot K in the 3-sphere S^3 is called an *L-space knot* if it admits a nontrivial Dehn surgery yielding an L-space, a rational homology sphere whose Heegaard Floer homology is as simple as possible [16]. It

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is a *positive* (resp. *negative*) L-space knot if a positive (resp. negative) Dehn surgery yields an L-space. In [9] Krcatovich proves that if a knot is a connected sum of non-trivial knots, then it is not an L-space knot. Since positive L-space knots are special types of tight fibered knots [5, 14], Theorem 1.1 allows us to generalize Krcatovich's result to non-trivial band sums:

Corollary 1.3. *If a knot in S^3 is a non-trivial band sum, then it is not an L-space knot.*

Since lens spaces, and more generally 3-manifolds with finite fundamental group, are L-spaces [16], Corollary 1.3 immediately implies:

Corollary 1.4. *Any knot obtained by a non-trivial band sum does not admit a finite surgery.*

Recall from [5, 1] that the set of strongly quasipositive, fibered knots coincides with that of tight fibered knots. Thus Theorem 1.1 says that any prime knot obtained by a non-trivial band sum fails to satisfy at least one of conditions (i) K is strongly quasipositive, and (ii) K is fibered.

In fact, we demonstrate:

Proposition 1.5. (1) *There exist tight fibered knots K_1 and K_2 , together with a band β , such that $K_1 \natural_{\beta} K_2$ is prime, strongly quasipositive, but not fibered.*
(2) *There exist tight fibered knots K_1 and K_2 , together with a band β , such that $K_1 \natural_{\beta} K_2$ is prime, fibered, but not tight (hence not strongly quasipositive).*

The connected sum of two tight fibered knots is tight fibered. On the other hand we show:

Proposition 1.6. *For any tight fibered knots K_1 and K_2 , there exists a band β such that $K_1 \natural_{\beta} K_2$ is fibered, but not tight (hence not strongly quasipositive).*

2. PROOFS OF THEOREM 1.1 AND COROLLARY 1.3

Proof of Theorem 1.1. Assume a knot $K = K_1 \natural_{\beta} K_2$ in S^3 is a non-trivial band sum of the split link $K_1 \sqcup K_2$ of knots K_1 and K_2 with a band β . Then $g(K) \geq g(K_1) + g(K_2)$ since genus is superadditive for band sums by Gabai [3, Theorem 1] and Scharlemann [19, 8.4 Theorem]. Since K is concordant to $K_1 \# K_2$ by Miyazaki [12, Theorem 1.1], $\tau(K) = \tau(K_1 \# K_2)$ where τ is the Ozváth-Szabó concordance invariant [15]. Furthermore, additivity of τ under connected sum [15, Proposition 3.2] shows $\tau(K_1 \# K_2) = \tau(K_1) + \tau(K_2)$. If we also suppose that $\tau(K) = g(K)$, then because τ gives a lower bound on genus for all knots in S^3 [15, Corollary 1.3], we will have the string of inequalities:

$$\tau(K) = g(K) \geq g(K_1) + g(K_2) \geq \tau(K_1) + \tau(K_2) = \tau(K).$$

It then follows that $g(K) = g(K_1) + g(K_2)$.

Now for the proposition, assume the band sum K is a tight fibered knot. It follows from Hedden [5, Proposition 2.1] and also Baader-Ishikawa [1, Theorem 3.1] that K is strongly quasipositive. Hence Livingston [11, Theorem 4] shows that $\tau(K) = g(K)$. It then follows from the above calculation that we have $g(K) = g(K_1) + g(K_2)$.

Now by Kobayashi [7, Theorem 2], both K_1 and K_2 are fibered, and the banding expresses K as the connected sum of K_1 and K_2 , i.e. there is a 2-sphere which split K_1 and K_2 and intersects β in a single arc. The proof of [7, Proposition 4.1] clarifies this last point. In particular, K cannot be prime. For otherwise, K_1 or K_2 would be a trivial knot and the band sum is trivial, a contradiction to the assumption. \square

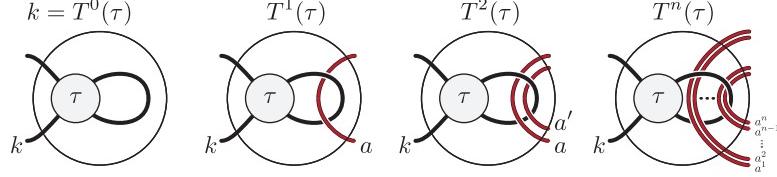


FIGURE 3.1. The tangles $k = T^0(\tau)$, $T = T^1(\tau) = (B, k \cup a)$, $T' = T^2(\tau) = (B, k \cup a \cup a')$, and $T^n(\tau) = (B, k \cup a^1 \cup a^2 \cup \dots \cup a^{n-1} \cup a^n)$.

Proof of Corollary 1.3. Let K be an L-space knot. By mirroring, we may assume it is a positive L-space knot. Then by combining Hedden and Ni [5, 14], K is a tight fibered knot. Krcatovich [10, 9] shows that K must be prime. Now Theorem 1.1 implies K cannot be a non-trivial band sum. Note that if K is a positive L-space knot, then the equality $\tau(K) = g(K)$ follows by [16]. \square

3. A FAMILY OF PRIME TANGLES

In Lemma 4.3 we show Example 4.2 of a band sum is prime by demonstrating that it has a prime tangle decomposition and then employing the work of Nakanishi [13]. The two tangles involved both have a similar form which we generalize to the family of $(n+1)$ -strand tangles $T^n(\tau)$ based off a two-strand tangle τ . (See Figure 3.1 and the discussion below.) In Proposition 3.3 we provide hypotheses that ensure the tangle $T^n(\tau)$ is prime for $n \geq 1$.

For our purposes, an *n-string tangle* is a proper embedding of a disjoint union of n arcs into a ball considered up to proper isotopy; n is a positive integer. Recall that a tangle T is *prime* if it satisfies the following conditions:

- (1) T is not a trivial 1-string tangle.
- (2) T is *locally trivial*: Any 2-sphere that transversally intersects T in just two points bounds a ball in which T is a trivial 1-string tangle.
- (3) T is *non-splittable*: Any properly embedded disk does not split T .
- (4) T is *indivisible*: Any properly embedded disk that transversally intersects T in a single point divides the tangle into two subtangles, at least one of which is the trivial 1-string tangle. In the following we say that such a disk is also *boundary-parallel*.

Given a two-strand tangle τ with fixed boundary so that the endpoints on the left belong to different arcs, define the $(n+1)$ -strand tangle $T^n(\tau)$ as in Figure 3.1 for non-negative integers n . For convenience, let us take $k = T^0(\tau) = (B, k)$, $T = T^1(\tau) = (B, k \cup a)$, and $T' = T^2(\tau) = (B, k \cup a \cup a')$ as also shown in Figure 3.1. Here B is the 3-ball.

Lemma 3.1. *If $T = T^1(\tau) = (B, k \cup a)$ is locally non-trivial, then k is a knotted arc and (B, τ) is locally non-trivial, containing a summand of the knotted arc k .*

Remark 3.2. This summand is not necessarily proper. For example, the tangle τ could be homeomorphic to a split tangle consisting of k and another arc.

Proof. Assume the tangle $(B, k \cup a)$ is locally non-trivial. Since a is a trivial arc, any local knotting would have to occur in k . Let S be a sphere bounding a ball B_S that intersects $k \cup a$ in a knotted subarc of k . By replacing S with a smaller one if necessary, we may assume the closure of the knotted 1-string tangle $(B_S, B_S \cap k)$ is a prime knot. Observe that the arc a together with an arc in ∂B bounds a disk δ that intersects k exactly once. We will inductively isotope S and k leaving a invariant so that $S \cap \delta = \emptyset$.

Assume that $S \cap \delta \neq \emptyset$. Because S is disjoint from a , $S \cap \delta$ is a collection of circles. A circle of $S \cap \delta$ that is innermost in δ bounds a subdisk $\delta' \subset \delta$.

If there is such a disk δ' that is disjoint from k , then the circle $\partial\delta'$ also bounds a subdisk $\sigma \subset S$ that is disjoint from k . Since $B - \mathcal{N}(k)$ is irreducible (because it is the exterior of the knot K , the closure of k) and $\delta' \cup \sigma \subset B - \mathcal{N}(k)$, we may isotope σ to δ' (in the complement of $k \cup a$) and then further isotope S to reduce $|S \cap \delta|$. Perform such isotopies until every circle of $S \cap \delta$ bounds a subdisk of δ that intersects k ; in particular, the circles $S \cap \delta$ are concentric circles in δ about the single intersection point $\delta \cap k$.

Now if $S \cap \delta \neq \emptyset$, then the innermost one in δ bounds a disk $\delta' \subset \delta$ that meets k in a single point and is contained in B_S . Then $\partial\delta'$ divides S into two disks σ and σ' that each intersect k once and δ' divides B_S into two balls bounded by spheres $\delta' \cup \sigma$ and $\delta' \cup \sigma'$. Since we have chosen S so that $(B_S, B_S \cap k)$ is a prime knot, one of these balls intersects k in a trivial arc. We may use these balls to isotope B_S along with $k \cap B_S$ into a collar neighborhood of δ' and then further to be disjoint from δ' (and δ).

Since $S \cap \delta = \emptyset$, B_S is contained in $(B - \mathcal{N}(\delta), k \cup a) \cong (B, \tau)$. Hence τ contains a locally knotted arc which is the summand of k sectioned off by B_S . \square

Proposition 3.3. *If k is a non-trivial arc without any proper summand in B , and τ is locally trivial, then $T^n(\tau)$ is prime for $n \geq 1$.*

Proof. We proceed to check the four conditions of primeness for $T^n(\tau)$:

(1) This is obvious since $T^n(\tau)$ is an $n + 1$ -string tangle and $n \geq 1$.

(2) If $T^n(\tau)$ were locally non-trivial, then since the arcs a^1, \dots, a^n are trivial arcs, the local knotting would have to occur in the component k . Thus the 2-string tangle $T = T^1(\tau) = (B, k \cup a)$ would also be locally non-trivial (where we let $a = a^1$). Then Lemma 3.1 shows that this implies that τ would have to be a locally non-trivial 2-string tangle containing a summand of k . This implies that τ is locally non-trivial, contradicting the assumption.

(3) Assume $T^n(\tau)$ is splittable. Then there is a disk D that separates one component of $T^n(\tau) = (B, k \cup a^1 \cup \dots \cup a^n)$ from another. Hence D must separate k from the arc a^i for some i . Since the arcs a^1, \dots, a^n are mutually isotopic in the complement of k , we may assume $k \cup a^1$ is splittable. However, this implies that $T^1(\tau) = (B, k \cup a^1)$ is locally non-trivial since k is a non-trivial arc, a contradiction to (2).

(4) Assume $T^n(\tau)$ is divisible. Then there is a disk D transversally intersecting $T^n(\tau)$ in just one point so that D is not boundary-parallel. Since the arcs a^1, \dots, a^n are mutually isotopic, we may assume either D intersects k or D intersects a^1 .

If D intersects k , then it must cut off a trivial arc from k since k has no proper summand. Because D is not boundary-parallel, some arc of a^1, \dots, a^n must be on this side, say a^i . But then $T^1(\tau) = (B, k \cup a^i)$ is locally non-trivial, a contradiction to (2).

If D intersects a^1 , then the arcs a^2, \dots, a^n and k must all be on the same side of D . Otherwise, or some $i \geq 2$, $T^1(\tau) = (B, k \cup a^i)$ would be a splittable 2-string tangle contrary to (3). But now since a^1 is a trivial arc, D must be boundary-parallel to the side that does not contain a^2, \dots, a^n and k . Again this is a contradiction. \square

4. EXAMPLES

In this section, we give examples that prove Propositions 1.5 and 1.6. Examples 4.2 and 4.4 give Proposition 1.5(1) and (2) respectively. Proposition 1.6 follows from Example 4.5.

4.1. A prime, non-fibered, strongly quasipositive banding of tight fibered knots. We show that there exists a banding $K = K_1 \natural_{\beta} K_2$ of tight fibered knots K_1 and K_2 which is prime and strongly

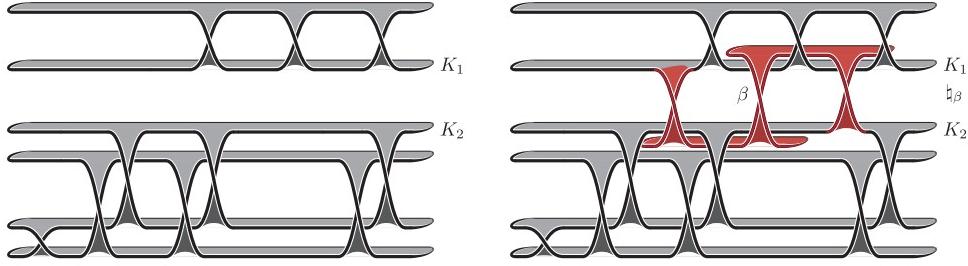


FIGURE 4.1. The $(2, 3)$ -torus knot K_1 and the $(2, 1)$ -cable of the $(2, 3)$ -torus knot K_2 are strongly quasipositive knots. They may be banded together to form another strongly quasipositive knot $K = K_1 \natural_\beta K_2$.

quasipositive; furthermore, K has the Alexander polynomial of an L-space knot. It follows from Theorem 1.1 (or by Kobayashi [7]) that K cannot be fibered, and thus it is not an L-space knot [14] (cf. Corollary 1.3).

Lemma 4.1. *If $K = K_1 \natural_\beta K_2$ and $g(K) = g(K_1) + g(K_2)$ then $\Delta_K(t) = \Delta_{K_1}(t)\Delta_{K_2}(t)$.*

Proof. By Gabai [3] and Scharlemann [19], when $g(K) = g(K_1) + g(K_2)$ for the band sum $K = K_1 \natural_\beta K_2$, then there are minimal genus Seifert surfaces F, F_1, F_2 for the knots K, K_1, K_2 such that F is the union of F_1, F_2 and the band β . Since F_1 and F_2 are separated by a sphere, F has a Seifert matrix that is block diagonal of the Seifert matrices for F_1 and F_2 . The result then follows. \square

Proposition 1.5(1) follows from the example below.

Example 4.2. Let K_1 be the $(2, 3)$ -torus knot $T_{2,3}$ and K_2 the $(2, 1)$ -cable of the $(2, 3)$ -torus knot $T_{2,3}^{2,1}$; K_2 wraps twice in a longitudinal direction and once in a meridional direction along the companion $T_{2,3}$. Note that K_1 and K_2 are both strongly quasipositive fibered knots, and hence tight fibered knots [5]. The left-hand side of Figure 4.1 shows the split link $K_1 \sqcup K_2$. The right hand side of Figure 4.1 shows the banding $K = K_1 \natural_\beta K_2$ which, by virtue of its presentation is also strongly quasipositive.

By Lemma 4.1, it now follows that $\Delta_K(t) = \Delta_{K_1}(t)\Delta_{K_2}(t) = (t - 1 + t^{-1})(t^2 - 1 + t^{-2})$. Indeed, let us note that we constructed K so that $\Delta_K(t)$ is also the Alexander polynomial of an L-space knot, namely the L-space knots $T_{3,4}$ and $T_{2,3}^{2,3}$, i.e. the $(3, 4)$ -torus knot and the $(2, 3)$ -cable of $(2, 3)$ torus knot.

Lemma 4.3 below uses tangle theory to shows that $K = K_1 \natural_\beta K_2$ is a prime knot. Hence by Theorem 1.1 (or by Kobayashi [7]), K is not fibered.

Lemma 4.3. *The knot $K = K_1 \natural_\beta K_2$ on the right-hand side of Figure 4.1 is a prime knot.*

Proof. Discarding the Seifert surface followed by a small isotopy presents $K = K_1 \natural_\beta K_2$ as on the left-hand side of Figure 4.2. The sphere separating K_1 and K_2 that intersects K in 6 points (shown first as a horizontal line) splits K into two 3-string tangles T_1 and T_2 as shown in Figure 4.2. Then Figure 4.3 shows, for each $i = 1, 2$, the results of further isotopies expressing T_i as the tangle $T^2(\tau_i) = k_i \cup a_i \cup a'_i$ for some 2-string tangle τ_i such that k_i is a knotted arc of knot type K_i while a and a' are trivial arcs that are isotopic in the complement of k_i . (See Figure 3.1.) Note that the knot K_i is the closure of τ_i while the arc $k(\tau_i)$ is the ‘half-closure’ of τ_i .

Since K_i is a non-trivial prime knot for each $i = 1, 2$, then the arcs k_i are non-trivial without any proper summand. Because τ_1 is a trivial tangle, it is locally trivial. Because k_2 has no proper summand, if τ_2 were locally non-trivial, the local knotting would have the knot type of k_2 . However since the two one-strand subtangles of τ_2 are a trivial arc and a knotted arc which one may identify as having the knot type of 8_{20} which is not the type of k_2 , τ_2 is locally trivial. (Figure 4.3 highlights one strand in each τ_1

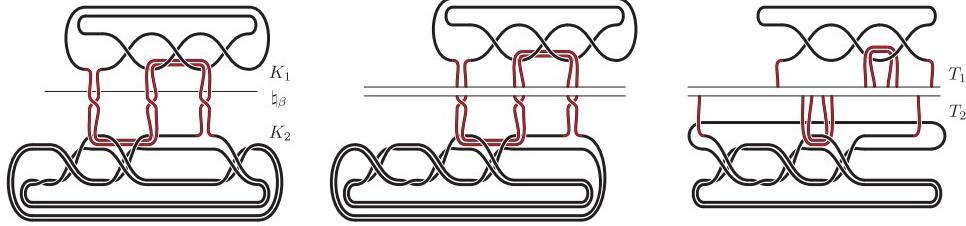


FIGURE 4.2. The splitting sphere of $K_1 \sqcup K_2$ divides $K = K_1 \natural_\beta K_2$ into two three strand tangles, T_1 and T_2 . The right-hand picture of the tangles gives isotopic versions in the center.

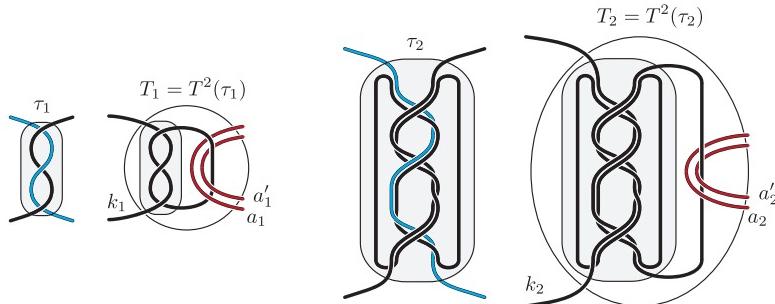


FIGURE 4.3. A further isotopy of the two tangles T_1 and T_2 exhibits them as the tangles $T^2(\tau_1)$ and $T^2(\tau_2)$.

and τ_2 so that the other strand is more easily discerned.) Thus Proposition 3.3 shows that T_1 and T_2 are both prime tangles. Work of Nakanishi then implies that K is a prime knot [13]. \square

4.2. A prime, fibered, non-strongly quasipositive banding of tight fibered knots. We show that there exists a banding $K = K_1 \natural_\beta K_2$ of tight fibered knots K_1 and K_2 which is prime, fibered, but not tight, and hence not strongly quasipositive.

Example 4.4. Take $K_1 = T_{2,3}$ and the trivial knot K_2 , each of which is a tight fibered knot. Let $K_1 \sqcup K_2$ be a split link, and β a band given in Figure 4.4. Then the band sum $K = K_1 \natural_\beta K_2$ is the prime knot 8_{10} . (This band sum presentation of 8_{10} was given by Kobayashi [8].) Recall that K is a Montesinos knot $M(\frac{1}{2}, \frac{1}{3}, \frac{2}{3})$ and it is easy to see that K is a plumbing of two torus knots $T_{2,5}$ and $T_{2,-3}$. Hence it is a fibered knot [20, Theorem 1]. By Theorem 1.1 K is not tight, and hence it is not strongly quasipositive.

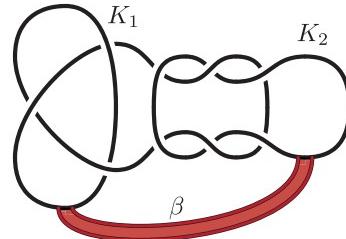


FIGURE 4.4. $K_1 = T_{3,2}$, K_2 = unknot, and $K = K_1 \natural_\beta K_2$ is 8_{10} .

4.3. A non-prime, fibered, non-strongly quasipositive banding of tight fibered knots. Recall that the connected sum of tight fibered knots is always tight fibered. In this subsection we show that for any tight fibered knots K_1 and K_2 , we can take a band β so that $K_1 \natural_\beta K_2$ is fibered, but not tight, and hence not strongly quasipositive.

Example 4.5. Figure 4.5 shows a band β for a split link of any two knots K_1 and K_2 so that the band sum $K = K_1 \natural_\beta K_2$ produces the connected sum $K = K_1 \# 3_1 \# \overline{3_1} \# K_2$. Since the square knot $3_1 \# \overline{3_1}$ is fibered, if K_1 and K_2 are fibered then K will be fibered. However since $3_1 \# \overline{3_1}$ is a non-trivial ribbon knot, $\tau(3_1 \# \overline{3_1}) = 0$, hence $\tau(K_1 \# 3_1 \# \overline{3_1} \# K_2) = \tau(K_1 \# K_2) \leq g(K_1 \# K_2) < g(K_1 \# 3_1 \# \overline{3_1} \# K_2)$. This shows K cannot be strongly quasipositive, and in particular it is not tight. Thus, choosing K_1 and K_2 to be tight fibered knots gives the example.

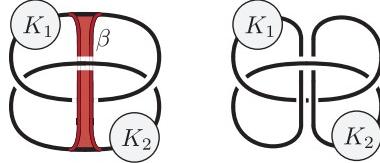


FIGURE 4.5. The band β for the split link $K_1 \sqcup K_2$ produces the band sum $K_1 \natural_\beta K_2 = K_1 \# 3_1 \# \overline{3_1} \# K_2$.

5. FURTHER DISCUSSION AND QUESTIONS

Our results and proofs lead to a few natural questions.

5.1. Band sums of strongly quasipositive knots.

Question 5.1. *If $K = K_1 \natural_\beta K_2$ is a strongly quasipositive knot, must K_1 and K_2 be strongly quasipositive?*

Since $\tau(K) = g(K)$ for strongly quasipositive knots, the argument in the proof of Theorem 1.1 enables us to conclude that $g(K) = g(K_1) + g(K_2)$. Then the equality implies that there are minimal genus Seifert surfaces F_1, F_2 for the knots K_1, K_2 that are disjoint from the band β so that $F = F_1 \cup \beta \cup F_2$ is a minimal genus Seifert surface for K ; see Gabai [3, Theorem 1] and Scharlemann [19, 8.5 Remark]. If F is a quasipositive Seifert surface, then it follows that F_1 and F_2 are also quasipositive Seifert surfaces (because they are subsurfaces of F) and hence K_1 and K_2 are strongly quasipositive knots.

From this point of view, an affirmative answer to this question would follow from an affirmative answer to the following.

Question 5.2. *If F is a minimal genus Seifert surface for a strongly quasipositive knot, must F be a quasipositive surface?*

Let us note that the Kakimizu complex for minimal genus Seifert surfaces for a knot is connected [6]. Thus if the answer to this question is negative, then there is a strongly quasipositive knot K with a quasipositive Seifert surface Q and a non-quasipositive minimal genus Seifert surface F such that $F \cap Q = \partial F = \partial Q = K$.

5.2. Band sums of split links. The band sum operation can be generalized as follows. Start with a split link with n components K_1, \dots, K_n (where for each component there is a sphere separating it from all of the other components), and connect the components via $n - 1$ pairwise disjoint bands to obtain a knot K . We call K a *band sum* of K_1, \dots, K_n ; see [12]. As an analogy of the case $n = 2$, we say that a band sum is *trivial* if K coincides with one of K_i ($1 \leq i \leq n$).

- Question 5.3.**
- (1) If a prime knot K in S^3 is a non-trivial band sum of K_1, \dots, K_n , then can K be a tight fibered knot?
 - (2) If a knot K in S^3 is a non-trivial band sum of K_1, \dots, K_n , then can K be an L-space knot?

We expect a negative answer to both of these questions. Since the relation $g(K_1) + \cdots + g(K_n) \geq \tau(K_1) + \cdots + \tau(K_n) = \tau(K)$ holds, if one follows the scheme of our proof of Theorem 1.1, it becomes a question of whether the “superadditivity of genus” for such band sums holds true and whether there is a result similar to Kobayashi’s work on fibering and band sums.

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